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1986 J. Phys. A: Math. Gen. 19 L1159

(http://iopscience.iop.org/0305-4470/19/18/008)

J. Phys. A: Math. Gen. 19 (1986) L1159–L1164. Printed in Great Britain

LETTER TO THE EDITOR

Spreading phenomena in which growth sites have a distribution of lifetimes

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Received 6 October 1986

Abstract. Originally, studies of the growth of fractal objects such as percolation clusters assumed that the growth sites have an infinite lifetime. Recently Bunde, Miyazima and Stanley have studied the effect of a fixed finite lifetime and they have found that the long-time growth evolves toward the kinetic growth walk with self-avoiding walk critical exponents. Here we consider for two dimensions the general case in which each growth site is randomly assigned infinite lifetime (with probability q) or a finite lifetime (with probability $1-q$). The phase diagram is similar to that of site-bond percolation, a model used to describe solvent effects in gelation.

Recently considerable attention has focused on various models of growth processes (see, e.g., the recent review by Herrmann [1] as well as recent conference proceedings [2–4]). This interest is motivated in part by the realisation that a wide variety of spreading phenomena—ranging from growth of epidemics and 'fires' to signal propagation and network mechanics—have features in common. These spreading phenomena have been studied by a family of growth models in which the perimeter of the growing object is partitioned into an active or 'growing' part and a 'blocked' part. The active perimeter sites are commonly called growth sites (perimeter sites are empty neighbours of cluster sites).

Perhaps the simplest growth model is that due to Eden [5], in which the entire perimeter is regarded as being active; growth proceeds by randomly choosing sites from this perimeter. The opposite extreme, in which almost the entire perimeter is blocked, is the kinetic growth walk (KGW) [6, 7]. Here the next site can be added only to the perimeter of the last added site. The KGW gives rise to clusters with the same statistics as those found for a simple self-avoiding random walk [8, 9].

An equivalent way of describing both the Eden model and the KGW is to think of the perimeter sites as being characterised by a lifetime τ during which cluster sites can infect or 'spread to' their so far uninfected neighbouring sites [10]. Thus each perimeter site is considered to be active only for τ time steps, and is inactive thereafter. The Eden model corresponds to $\tau=\infty$ and the KGW to $\tau=1$. More generally, for a fixed lifetime τ_0 between 1 and ∞ (figure 1(a)), the cluster initially resembles an Eden cluster but after a characteristic time $t_\times$ it crosses over to its asymptotic (large-time) form, a branched object which, numerically, has the same fractal dimension d_f as the KGW ($d_f=\frac{4}{3}$ for $d=2$) [10].

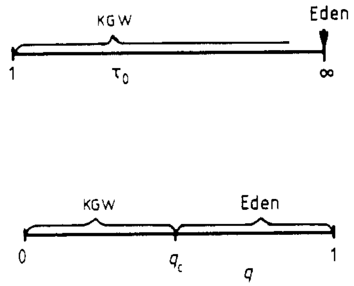

Figure 1. (*a*) Phase diagram for the growth model of [10] in which each growth site is assumed to remain active only for a time τ_0. For $\tau_0=\infty$, one finds Eden clusters ($d_f=2$). For finite values of τ_0, one finds Eden clusters initially, but there is a crossover at large time to the asymptotic result, a KGW ($d_f=\frac{4}{3}$, $d_g=0$). (*b*) Phase diagram of the present growth model, in which a non-zero fraction q of the growth sites is assumed to remain active forever, with the remaining fraction $(1-q)$ becoming inactive after time τ_0. One finds Eden clusters for all values of q above a critical value q_c (which turns out to be the bond percolation threshold). Below q_c one finds KGW behaviour if $\tau_0>0$ and finite clusters only if $\tau_0=0$.

In previous work, all growth sites were treated as having the same lifetime. Motivated by the desire to describe spreading phenomena (such as model 'fires' in inhomogeneous media) in which growth sites are not all identical (e.g. not all trees in the front of the fire burn exactly τ time units), we introduce here a model in which growth sites have a heterogeneity of lifetimes. We find that, even for strongly singular distributions,

$$P(\tau)\sim\tau^{-(1+\alpha)}\qquad \alpha>0 \tag{1}$$

the growth process is characterised by KGW exponents. This finding is not surprising since KGW exponents describe the homogeneous lifetime case for all finite values τ_0. In order to change the universality class, we must allow a non-zero fraction of the growth sites to have infinite lifetime. Indeed, we find strikingly different behaviour for a model in which a fraction q of the growth sites have infinite lifetime, with the remaining fraction $1-q$ having a finite lifetime τ_0, i.e.

$$\tau=\begin{cases}\infty & \text{probability } q\\ \tau_0 & \text{probability } 1-q.\end{cases} \tag{2}$$

This new model reduces to the model discussed recently by Bunde *et al* [10] when $q=0$. We find that at a critical value of the parameter q, q_c, the exponents change from KGW to Eden (figure 1(*b*)) for $\tau_0>0$. The special case $\tau_0=0$ gives rise to only finite clusters below q_c. Our finding will be discussed below in terms of site-bond percolation, a polyfunctional condensation model used to describe solvent effects on gelation [11-15], where q plays the role of the bond probability p_b and q_c is the bond percolation threshold ($q_c=\frac{1}{2}$ for the square lattice studied here).

It is also possible to block growth sites from the outset. Suppose, e.g., that a fraction $1-p$ of the growth sites of the Eden model ($\tau=\infty$) are blocked. Then for p small, one still generates the compact Eden clusters. However, for a critical value $p=p_c$ given by the site percolation threshold, one finds the incipient percolating cluster [16-18]. Below p_c, only finite clusters exist.

We generalised our model to include this possibility that a fraction $1-p$ of growth sites are blocked from the outset. We found (figures 2(*a*) and (*b*)) that the p-q phase diagram is characterised by a phase boundary or line of percolation critical points bounding a region of Eden exponents. This part of the phase diagram is similar to the site-bond percolation model of the sol-gel transition (figure 2(*c*)), an analogy we shall elaborate upon below.

To obtain the phase diagram, we first calculate $\langle r^2 \rangle$, the mean square distance of the last added site from the origin. Since a new site is added at each unit of time t, the cluster mass is given by t and, asymptotically,

$$\langle r^2 \rangle \sim t^{2/d_f}. \tag{3}$$

We also studied how G, the number of growth sites, scales as

$$G \sim t^{d_g/d_f} \tag{4}$$

where d_g is the fractal dimension *of the growth sites*. From (3) and (4) we thus obtain d_f and d_g.

In the following, we discuss the two cases $\tau_0 = 0$ and $\tau > 0$ separately.

(*a*) $\tau_0 = 0$. For $\tau_0 = 0$, the present growth model reduces, in the asymptotic ($t \to \infty$) limit, to an *equilibrium* (non-growth) model, site-bond percolation, which was proposed in order to describe the phenomenon of polyfunctional condensation of f-functional monomers in a solvent [11-15]. The monomers and solvent molecules are considered to be randomly distributed on a lattice, with p being the probability that a site is occupied by a monomer and $1-p$ the probability that it is occupied by a solvent molecule. Neighbouring sites occupied by monomers can be chemically bonded with a probability p_b. Thus if $p_b = 1$, site-bond percolation reduces to pure site percolation, with a critical point $p_s^* \simeq 0.593$ (for the square lattice) separating a region of finite clusters ('sol phase') from a region in which an infinite cluster is present ('gel phase'). Similarly, if $p_s = 1$, site-bond percolation reduces to pure bond percolation with a

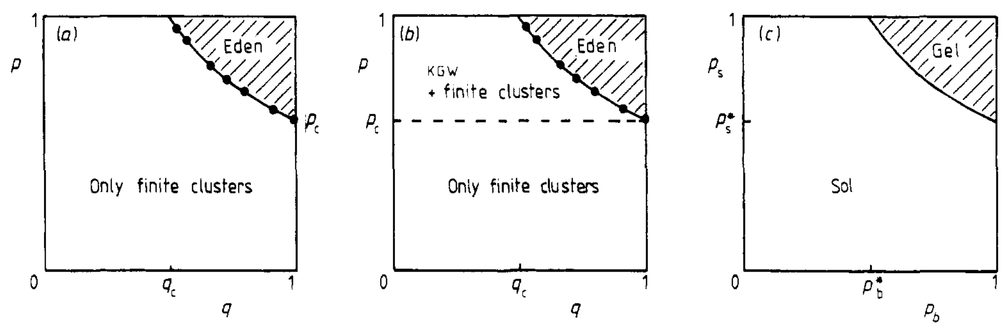

Figure 2. (*a*) Generalisation of the phase diagram of figure 1(*b*) to the case in which growth sites are open with probability p and blocked with probability $1-p$ and $\tau_0 = 0$. There is a critical line separating (i) a region where one finds Eden clusters and (ii) a region where one finds finite clusters. (*b*) Same as (*a*), except $\tau_0 > 0$. The horizontal broken line separates a finite cluster regime for $p < p_c$ from a regime for $p \geq p_c$ in which there is, in addition to finite clusters, a single infinite cluster on which KGW can grow. (*c*) Analogous phase diagram for the site-bond model of polyfunctional condensation of f-functional monomers in a solvent. Again, there is a critical line separating (i) a region where one finds an infinite cluster or 'gel molecule' and (ii) a region where one finds only finite clusters (the 'sol phase').

critical point $p_b^* = \frac{1}{2}$ separating the sol from the gel phase. The general phase diagram is shown in figure 2(*c*).

Clearly the parameter p of the present model plays the same role as p_s in site-bond percolation: clusters in the sol phase can grow only if a site is unblocked and this occurs with probability p. The parameter q of the present model plays the same role as p_b in site-bond percolation. To see this, consider first the case $p = 1$, so that none of the sites is blocked from the outset. A fraction q of the growth sites has infinite lifetime and a remaining fraction $1 - q$ has lifetime zero (they become inactive immediately). That is, of the z bonds that the epidemic can use to spread at a given time t, a fraction q are open and a remaining fraction $1 - q$ are blocked. Thus q plays the role of the bond probability.

Another way of introducing site-bond percolation is to start with the fully occupied case $p_s = p_b = 1$ (all sites occupied by monomers, all bonds intact) and $d_f = d$ ('gel phase'). Then we can perturb away from this limit by replacing active monomers by inactive solvent molecules ($p_s \leqslant 1$). When only a few monomers are removed, we still have the gel phase but when a critical solvent concentration is reached, the gel phase can no longer exist and we have a critical point, below which the sol phase exists. Similarly, if we perturb away from the limit by removing bonds ($p_b \leqslant 1$), we find a critical point (the bond percolation threshold) below which we have only finite clusters.

The present model can be regarded in an analogous fashion. We start with the Eden model ($p = 1$, $q = 1$): all sites are present, growth sites live forever and $d_f = d$ and $d_g = d - 1$. If only a few growth sites are blocked ($p \leqslant 1$), we still have the Eden model but when a critical concentration is reached, we find percolation clusters. Similarly, if we perturb away from the limit $p = q = 1$ by allowing a fraction $(1 - q)$ of the growth sites to have zero lifetime, then we find a critical point (the bond percolation threshold p_b^*) below which only finite clusters can exist. We have confirmed this picture by numerical simulations.

(*b*) $\tau_0 > 0$. For $\tau_0 > 0$, we find numerically that the present model has the same critical line (with percolation exponents $d_f = \frac{91}{48}$ and $d_g \simeq \frac{3}{4}$) bounding the region of compact Eden clusters ($d_f = 2$). However, the phase diagram outside the Eden region differs, since for $p > p_c$ *clusters can be grown that span the system*. We anticipate finding KGW exponents since we know [10] from figure 1(*a*) that KGW exponents are found along the vertical line $q = 0$ of figure 2(*b*); adding a small fraction q of infinite lifetime sites should not change the asymptotic behaviour. To test this idea (and to determine the phase diagram) we carried out an extensive series of Monte Carlo simulations, using lattices of size 800×800 and t up to 20 000 time steps.

Figure 3 shows log-log plots of $\langle r^2 \rangle$ against time for three representative sets of (q, p), corresponding to the three distinct regions of the phase diagram of figure 2(*a*) where infinite clusters can grow; $\tau_0 = 3$ for all plots. The data for the KGW region of the phase diagram have slope $2/d_f = \frac{3}{2}$, corresponding to the fact that $d_f = \frac{4}{3}$ for the KGW. The data for $p = p_c$ have slope $2/d_f = 1.05$, since $d_f = \frac{91}{48}$ for percolation. Finally, the data for the Eden region of the phase diagram have slope $2/d_f = 1$.

Figure 4 shows the corresponding log-log plots of G against time for the same parameter values as shown in figure 3. We find $d_g = 0$ in the KGW region (since asymptotically the number of growth sites approaches a constant). At the phase boundary, we find the percolation value $d_g/d_f \simeq 0.4$ [16-18]. In the Eden region d_g approaches 1.

In summary, then, we have introduced a dynamic growth model suitable for describing phenomena such as model forest fires and epidemics where the growth sites

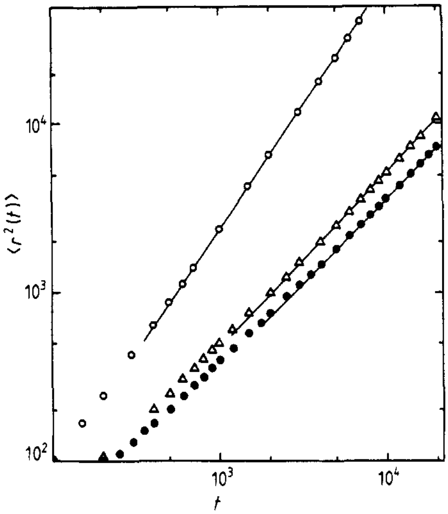

Figure 3. Log-log plot showing the dependence on time t of $\langle r^2\rangle$, the mean square distance of the last added site from the seed. For representative points from the three regions of the phase diagram of figure 2(a): (○) $p=0.8$, $q=0.5$ (the KGW region), (△) $p=1$, $q=0.5$ (the critical line) and (●) $p=0.9$, $q=0.9$ (the Eden region). The asymptotic slope gives $2/d_f$. The points shown represent averages of more than 5000 configurations for each time step. The lattice size is 800×800 and no configuration touched the boundary.

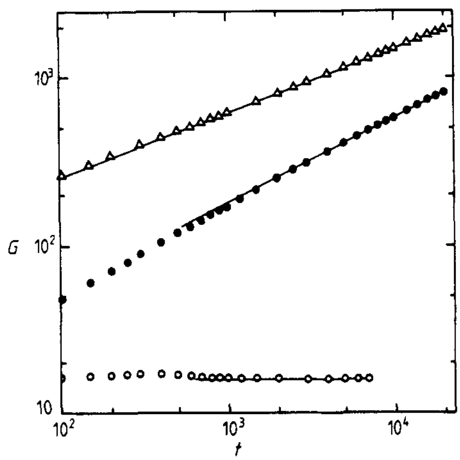

Figure 4. Log-log plot showing the dependence on time t of G, the number of active growth sites, for the same three representative points from the three regions of the phase diagram shown in figure 3. The asymptotic slope is d_g/d_f.

can have a range of lifetimes. We find that the same fractal structures are generated *unless* the distribution of lifetimes has a component with infinite lifetime. We carried out a series of Monte Carlo simulations for a model in which a fraction q of the sites are present with lifetime $\tau=\infty$, the remaining fraction $1-q$ are present with lifetime τ_0 and found that for $\tau_0=0$ the phase diagram for the asymptotic $t\to\infty$ limit is *identical* to the site-bond percolation model of polyfunctional condensation of f-functional monomers in a solvent. For $\tau_0>0$, we find an additional phase, characterised by KGW exponents ($d_f=\frac{4}{3}$, $d_g=0$) for that region of the phase diagram with $p\geqslant p_c$.

This work has been supported by a grant from Chubu University as well as by Deutsche Forschungsgemeinschaft, by Minerva, and by the USA-Israel BiNational Science Foundation. The Center for Polymer Studies is supported by grants from the NSF and the ONR.

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